SOME INEQUALITIES FROM "COUNTING ZEROS OF DIRICHLET L-FUNCTIONS" Thursday 23rd April, 2020 18:08

MICHAEL A. BENNETT, GREG MARTIN, KEVIN O'BRYANT, AND ANDREW RECHNITZER

ABSTRACT. We give some details on how we prove the inequalities in "Counting Zeros of Dirichlet L-Functions".

CONTENTS

1. INEQUALITIES FROM THE MAIN BODY OF THE PAPER

1.1. $g(a,T)$.

Lemma 1.1. *Let $a \in \{0,1\}$ and $T \geq \frac{5}{7}$. Then*

$$|g(a,T)| \leq \frac{2-a}{50T}.$$

1.2. $E(a,d,T)$.

$$\frac{E(a,d,T)}{\pi} \leq \frac{(a+8)d^2}{15(2a+3T-1)} + \frac{1}{128}$$

2. THE MAIN BOUND: LARGE ℓ

In this section, we work under the assumption that $\ell \geq 27 + \frac{1}{50} = \frac{1351}{50}$ and $T \geq \frac{5}{7}$, and $a \in \{0,1\}$. Further, define our fundamental parameters in terms of ℓ as follows:

$$\eta := \frac{18}{10+9\ell} \subseteq \left(0, \frac{900}{12659}\right] \subseteq (0, 0.08)$$

$$c := 1 + \frac{391}{683+74\ell} \subseteq \left(1, \frac{76837}{67062}\right] \subseteq (1, 1.15)$$

$$r := \frac{149}{140} + \frac{769}{512+30\ell} \subseteq \left(\frac{149}{140}, \frac{1523637}{925820}\right] \subseteq (1.06, 1.65).$$

From these, we have several further expressions we will make use of:

$$\sigma_1 := c + \frac{(c-1/2)^2}{r} \subseteq \left(\frac{184}{149}, \frac{4793676918843499}{3426135363028314}\right] \subseteq (1.23, 1.40)$$

$$\delta := 2c - \sigma_1 - \frac{1}{2} \subseteq \left(\frac{79}{298}, \frac{672158744063911}{1713067681514157}\right] \subseteq (0.26, 0.40)$$

We have the quantity (which we care about more than anything)

$$\left| N(T,\chi) - \left(\frac{T}{\pi}\log\frac{qT}{2\pi e} + b_\chi\right)\right| \tag{2.1}$$

2010 *Mathematics Subject Classification.* Primary 11N13, 11N37, 11M20, 11M26; secondary 11Y35, 11Y40.

bounded above by

$$|g(a,T)| + \frac{E_\delta}{\pi} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{\log\zeta(c)/\zeta(2c)}{\log r/(c-1/2)} + \frac{1}{\log r/(c-1/2)}\cdot\frac{1}{\pi}\int_0^\pi F_{c,r}(\theta)\,d\theta$$

From earlier in this document, we have

$$|g(a,T)| \le \frac{2-a}{50T} \text{ and } \frac{E_\delta}{\pi} \le \frac{(a+8)d^2}{15(2a+3T-1)} + \frac{1}{128}.$$

We combine those, and then maximize over a, to get the following lemma.

Lemma 2.1. $|g(a,T)| + \frac{E_\delta}{\pi} \le \frac{1}{128} + \frac{17/250}{T-1/5}$.

Proof. The lower bound on $\frac{1}{128} + \frac{17/250}{T-1/5} - |g(a,T)| - E(a,d,T)/\pi$ of

$$\frac{1}{128} + \frac{17/250}{T-1/5} - \frac{2-a}{50T} - \frac{(a+8)d^2}{15(2a+3T-1)} - \frac{1}{128}$$

is clearly monotone in $d \in [79/298, 672158744063911/1713067681514157]$. For each value of a and d, it is a rational expression in a single variable T, and so we can minimize it using basic calculus (and rigorous root bounding for polynomials). We find that it is nonnegative. □

Lemma 2.2.

$$\frac{2}{\pi}\log\zeta(\sigma_1) - \frac{\log\zeta(2c)}{\log r/(c-1/2)} \le \frac{3(96161\ell - 3150232)}{256\,(128\ell^2 + 9637\ell + 164296)} + \frac{89}{256}$$

Proof. We note that σ_1 and $\log r/(c-1/2)$ are both monotone in ℓ. Also, the RHS is rational in one variable ℓ. This allows us to create a sufficiently precise interval extension of

$$\frac{3(96161\ell - 3150232)}{256\,(128\ell^2 + 9637\ell + 164296)} + \frac{89}{256} - \left(\frac{2}{\pi}\log\zeta(\sigma_1) - \frac{\log\zeta(2c)}{\log r/(c-1/2)}\right).$$

Splitting the domain $[27 + 1/50, \infty]$ into 989 subintervals allows us to prove its nonnegativity. □

It remains to bound

$$\frac{\log\zeta(c)}{\log r/(c-1/2)} + \frac{1/\pi}{\log r/(c-1/2)}\int_0^\pi F_{c,r}(\theta).$$

Lemma 2.3. *With the settings for c, r, η and bound on ℓ in this section, we have*

$$-\frac{1}{2} < c - r < 1 - c < -\eta < 0 < 1 < 1 + \eta < c < \sigma_1 < c + r.$$

With these inequalities, we can resolve θ_σ for the subscripts σ of interest:

$$\theta_{1-c} = \arccos\frac{1-2c}{r}, \theta_{-\eta} = \arccos\frac{-\eta - c}{r}, \theta_{1+\eta} = \arccos\frac{1+\eta-c}{r}, \theta_{-1/2} = \pi.$$

We set

$$J_1 = 64, J_2 = 24,$$

and take note of the expressions

$$\kappa_1 := (\theta_{-\eta} - \theta_{1+\eta}) \frac{1+\eta-c}{2} - (\pi - \theta_{-\eta}) \left(c - \frac{1}{2}\right) + \frac{r\left(\sin\theta_{-\eta} + \sin\theta_{1+\eta}\right)}{2}$$

$$\kappa_2 := \frac{\pi}{4J_1}\left(\log\zeta(c+r) + 2\sum_{j=1}^{J_1-1} \log\zeta(c + r\cos\tfrac{\pi j}{2J_1})\right)$$

$$\kappa_3 := \frac{\pi - \theta_{1-c}}{2J_2}\left(\log\zeta(1-c+r) + 2\sum_{j=1}^{J_2-1} \log\zeta\big(1-c-r\cos(\theta_{1-c} + (\pi-\theta_{1-c})\tfrac{j}{J_2})\big)\right)$$

$$\kappa_7 := \frac{1}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} (1+\eta-\sigma)\left(2t - 4 + \tfrac{7}{19}\left((1+\sigma)^2 + (t-2)^2\right)\right)\, d\theta$$

$$\kappa_8 := \frac{1}{4}\int_{\theta_{-\eta}}^{\pi} (1-2\sigma)\left(2t - 4 + \tfrac{7}{19}\left((1+\sigma)^2 + (t-2)^2\right)\right)\, d\theta.$$

Using results from the paper, we now have the bound

$$\begin{aligned}\frac{1/\pi}{\log r/(c-1/2)}\Bigg(&\pi\zeta(c) + \kappa_1\ell + +(\theta_{-\eta} - \theta_{1+\eta})\log\zeta(1+\eta) \\ &+ \int_0^{\theta_{1+\eta}} \log\zeta(\sigma)\, d\theta + \int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)\, d\theta \\ &+ \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{4} L_1(\theta)\, d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{4} L_1(\theta)\, d\theta\Bigg).\end{aligned}$$

which in turn is bounded by (setting $J_1 = 64$ and $J_2 = 24$)

$$\begin{aligned}\frac{1/\pi}{\log r/(c-1/2)}\Bigg(&\pi\zeta(c) + \kappa_1\ell + (\theta_{-\eta} - \theta_{1+\eta})\log\zeta(1+\eta) \\ &+ \frac{\log\zeta(1+\eta) + \log\zeta(c)}{2}(\theta_{1+\eta} - \frac{\pi}{2}) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_2 \\ &+ \frac{\log\zeta(1+\eta) + \log\zeta(c)}{2}(\theta_{1-c} - \theta_{-\eta}) + \frac{\pi - \theta_{1-c}}{2J_2}\log\zeta(c) + \kappa_3 \\ &+ \frac{\kappa_7}{T+2} + \frac{\kappa_8}{T+2}\Bigg).\end{aligned}$$

We address the main term, and most difficult term, first.

Lemma 2.4.

$$\frac{1/\pi}{\log r/(c-1/2)} \cdot \kappa_1\ell \leq \frac{399}{256} + \frac{930529\ell - 14168929}{256\left(32\ell^2 + 3105\ell + 38735\right)} + \frac{238413}{2^{20}}\ell \tag{2.2}$$

Proof. First, we note that

$$\frac{1/\pi}{\log r/(c-1/2)} \cdot \kappa_1\ell \sim \left(\frac{\frac{5}{7} - \frac{1}{2}\arccos\frac{140}{149}}{\pi\log\frac{149}{70}}\right)\ell < \frac{238413}{2^{20}}\ell,$$

so that the bound works asymptotically.

Dividing through by ℓ, we need to show that

$$\frac{798\ell^2 + 135589\ell + 80396}{\ell\left(512\ell^2 + 49680\ell + 619760\right)} + \frac{238413}{2^{20}} - \frac{\kappa_1}{\pi\log\frac{r}{c-1/2}} \geq 0.$$

To get a sufficiently efficient interval enclosure, we note that $\frac{798\ell^2+135589\ell+80396}{\ell(512\ell^2+49680\ell+619760)} + \frac{238413}{2^{20}}$ is monotone, as is $\frac{1}{\pi\log\frac{r}{c-1/2}}$. Within κ_1, the terms

$$\theta_{-\eta} - \theta_{1+\eta}, \frac{1+\eta-c}{2}, \pi - \theta_{-\eta}, c - \frac{1}{2}$$

are each monotone. The sines term simplifies as

$$r(\sin\theta_{-\eta} + \sin\theta_{1+\eta}) = \sqrt{r^2 - (c+\eta)^2} + \sqrt{r^2 - (c-1-\eta)^2}$$

and is also seen (with calculus) to be monotone.

Now, breaking the interval $[27 + 1/50, \infty]$ up into 73813 subintervals, we have computed nonnegativity. □

We collect the $\log\zeta(c)$ terms.

Lemma 2.5.

$$\frac{\log\zeta(c)}{2\log\frac{r}{c-1/2}}\left(\frac{3}{2} + \frac{1}{J_2} + \frac{1}{2J_1} + \frac{1}{\pi}((1-1/J_2)\theta_{1-c} + \theta_{1+\eta} - \theta_{-\eta})\right) \le \frac{640174922 - 24717757\ell}{512\,(512\ell^2 + 75117\ell + 496726)} + \frac{\ell}{2^{20}} + \frac{1365}{1024}\log(1+\ell) - \frac{1135}{512}$$

Proof. Notice that the $\ell/2^{20}$ term is asymptotically too large, but is included to make the computations tractable.

As before, we move all terms to the right side and need to show nonnegativity. We need two additional "primitive" functions to allow the interval analysis to go forward; in particular, to allow the $\log(1+\ell)$ terms to absorb the $-\log\zeta(c)$ terms. We define

$$LZPL(s) := \log\zeta(s) + \log(s-1), LZPL(1) = 0$$

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ontinuous and monotone increasing for $s \ge 1$ by an email from Greg. Also, we define for $s \ge 1$

$$LOL(s,\ell) := \frac{\log(s+\ell)}{\log(1+\ell)},$$

which is monotone decreasing by calculus.

As before, all of

$$\theta_{-1-\eta}, \theta_\eta, \theta_{1-c}, \log\frac{r}{c-1/2}$$

are monotonic. And while

$$\frac{640174922 - 24717757\ell}{512\,(512\ell^2 + 75117\ell + 496726)} + \frac{\ell}{1048576} - \frac{1135}{512}$$

is not montonic, it is rational in the single variable ℓ, and so we can maximized/minimized on any interval using calculus and rigorous bounds on roots of polynomials with one variable (it has a single critical value, located between 99 and 100).

For the interesting terms, we have

$$\log\zeta(c) = LZPL(c) + \left(LOL(\frac{683}{74},\ell) - \frac{\log 391/74}{\log(1+\ell)}\right)\log(1+\ell)$$

whence

$$\frac{1365}{1024}\log(1+\ell) - \log\zeta(c)f(\ell) = \left(\frac{1365}{1024} + f(\ell)\left(\frac{\log 391/74}{\log(1+\ell)} - LOL(\frac{683}{74},\ell)\right)\right)\log(1+\ell) - f(\ell)LZPL(c)$$

Breaking the domain $[27 + 1/50, \infty]$ into 717 pieces, we prove nonnegativity. □

We now collect all of the $\log\zeta(1+\eta)$ terms.

Lemma 2.6.

$$\frac{\log\zeta(1+\eta)}{2\log\frac{r}{c-1/2}}\left(\frac{\theta_{-\eta} + \theta_{1-c} - \theta_{1+\eta}}{\pi} - \frac{1}{2}\right) \le -\frac{3(7328646\ell - 24938773)}{2^8\,(512\ell^2 + 91562\ell + 599789)} + \frac{\ell}{2^{22}} + \frac{529}{2^{10}}\log(1+\ell) - \frac{91}{2^8}$$

Proof. As with with $\log\zeta(c)$ terms, we provide a bound that is not asymptotically sharp to ease the computational burden. We proceed exactly as we did with the $\log\zeta(c)$ terms, needing just 172 subintervals to prove nonnegativity. □

Lemma 2.7.

$$\frac{\kappa_2/\pi}{\log r/(c-1/2)} \le \frac{635}{1024} - \frac{9(113745\ell + 25384532)}{64\,(512\ell^2 + 150141\ell + 7149852)}$$

Proof. We note that $(9(25384532 + 113745\ell))/(64(7149852 + 150141\ell + 512\ell^2))$ is monotonic. Otherwise, the natural enclosure suffices, with 369 subintervals. ☐

Lemma 2.8.
$$\frac{\kappa_3/\pi}{\log r/(c-1/2)} \le \frac{491}{1024} - \frac{3346893\ell + 33179656}{512\,(512\ell^2 + 21113\ell + 208616)}$$

Proof. The following are all monotone:
$$\log\frac{r}{c-1/2}, \frac{3346893\ell + 33179656}{512\,(512\ell^2 + 21113\ell + 208616)}, \theta_{1-c}, 1-c+r.$$
Using 473 intervals is then sufficient. ☐

Lemma 2.9.
$$\frac{\kappa_7/\pi}{\log r/(c-1/2)} \le \frac{1237834\ell + 24697311}{256\,(512\ell^2 + 63962\ell + 800695)} - \frac{25}{256}.$$

Proof. We introduce the shorthand
$$\begin{aligned}
\alpha_1 &:= \sqrt{r^2 - (c-\eta-1)^2}\\
\alpha_2 &:= \sqrt{r^2 - (c+\eta)^2}\\
\beta_1 &:= 14c^2 - 7c\eta + 14c - 7\eta + 7r^2 - 48\\
\beta_2 &:= -14c^2 + 21c\eta - 7c + 21\eta - 7r^2 + 55\\
\beta_3 &:= -7c^2\eta + 7c^3 + 7c^2 - 14c\eta + 14cr^2 - 55c + 41\eta - 7\eta r^2 + 41,
\end{aligned}$$
for five expressions that monotonic in ℓ. Then
$$76\kappa_7 = \beta_3\left(\arctan\frac{c-\eta-1}{\alpha_1} - \arctan\frac{c+\eta}{\alpha_2}\right) + \alpha_1\beta_1 + \alpha_2\beta_2 + 5(1+2\eta)^2$$
by the fundamental theorem of calculus.

Dividing into 2213 subintervals completes the computation of nonnegativity. ☐

Lemma 2.10.
$$\frac{\kappa_8/\pi}{\log r/(c-1/2)} \le \frac{-1528043\ell - 177665980}{256\,(512\ell^2 + 113665\ell + 3255348)} - \frac{21}{256}$$

Proof. We introduce the shorthand
$$\begin{aligned}
\alpha_2 &:= \sqrt{r^2 - (c+\eta)^2}\\
\alpha_3 &:= \arccos\left(-\frac{c+\eta}{r}\right)\\
\beta_1 &:= -14\pi c^3 + (10-21\pi)c^2 + c\left(-28\pi r^2 - 20r + 96\pi - 10\right) - 7\pi r^2 + 10r^2 + 10(r - \eta(\eta+1)) - 41\pi\\
\beta_2 &:= 2\left(7c(2c-\eta+2) - 7\eta + 7r^2 - 48\right)\\
\beta_3 &:= c\left(7c(2c+3) + 28r^2 - 96\right) + 7r^2 + 41
\end{aligned}$$
for five expressions that monotonic in ℓ. The fundamental theorem of calculus gives us
$$76\kappa_8 = \beta_1 + \beta_2\alpha_2 + \beta_3\alpha_3.$$
Finally,
$$\frac{-1528043\ell - 177665980}{256\,(512\ell^2 + 113665\ell + 3255348)} - \frac{21}{256}$$
is also monotone. We now have an interval enclosure for
$$\frac{-1528043\ell - 177665980}{256\,(512\ell^2 + 113665\ell + 3255348)} - \frac{21}{256} - \frac{\beta_1 + \beta_2\alpha_2 + \beta_3\alpha_3}{76\log r/(c-1/2)}$$
that is efficient enough, and 10327 subintervals allow us to complete the verification. ☐

We now have brought our bound on (2.1) to the form

$$\begin{aligned}
&\frac{1}{128}+\frac{17/250}{T-1/5}+\frac{3(96161\ell-3150232)}{256\,(128\ell^2+9637\ell+164296)}+\frac{89}{256}+\frac{399}{256}+\frac{930529\ell-14168929}{256\,(32\ell^2+3105\ell+38735)}+\frac{238413}{2^{20}}\ell\\
&\quad+\frac{640174922-247177757\ell}{512\,(512\ell^2+75117\ell+496726)}+\frac{\ell}{2^{20}}+\frac{1365}{1024}\log(1+\ell)-\frac{1135}{512}-\frac{3(7328646\ell-24938773)}{2^8\,(512\ell^2+91562\ell+599789)}\\
&\qquad+\frac{\ell}{2^{22}}+\frac{529}{2^{10}}\log(1+\ell)-\frac{91}{2^8}+\frac{635}{1024}-\frac{9(113745\ell+25384532)}{64\,(512\ell^2+150141\ell+7149852)}\\
&\qquad\qquad+\frac{491}{1024}-\frac{3346893\ell+33179656}{512\,(512\ell^2+21113\ell+208616)}\\
&\qquad+\frac{1}{T+2}\left(\frac{1237834\ell+24697311}{256\,(512\ell^2+63962\ell+800695)}-\frac{25}{256}+\frac{-1528043\ell-177665980}{256\,(512\ell^2+113665\ell+3255348)}-\frac{21}{256}\right)
\end{aligned}$$

Collecting like terms, we have

$$\frac{953657}{2^{22}}\ell+\frac{947}{512}\log(1+\ell)+\frac{113}{256} \tag{2.3}$$

$$+\frac{17/250}{T-1/5}+\frac{2^{-8}}{T+2}\left(\frac{1237834\ell+24697311}{512\ell^2+63962\ell+800695}-\frac{1528043\ell+177665980}{512\ell^2+113665\ell+3255348}-46\right) \tag{2.4}$$

$$+\frac{1}{2^9}\left(\frac{576966\ell-18901392}{128\ell^2+9637\ell+164296}+\frac{1861058\ell-28337858}{32\ell^2+3105\ell+38735}+\frac{-3346893\ell-33179656}{512\ell^2+21113\ell+208616}\right. \tag{2.5}$$

$$\left.+\frac{640174922-247177757\ell}{512\ell^2+75117\ell+496726}+\frac{149632638-43971876\ell}{512\ell^2+91562\ell+599789}+\frac{-8189640\ell-1827686304}{512\ell^2+150141\ell+7149852}\right). \tag{2.6}$$

The terms on line (2.4) are monotone in ℓ, providing an interval enclosure for the terms on that line. Breaking the domain $\ell\in[27+\frac{1}{50},\infty], T\in[\frac{5}{7},\infty]$ into just 19 two-dimensional pieces, we find that the terms on this line are bounded by $\frac{7}{106}$. This results in a bound that depends on just one variable.

To show that this is bounded by $\frac{22737}{10^5}\ell+2\log(1+\ell)-\frac{682}{1000}$, we subtract, giving

$$\begin{aligned}
&\frac{5939\ell}{13107200000}+\frac{77}{512}\log(1+\ell)-\frac{2017297}{1696000}\\
&\quad+\frac{1}{2^9}\left(\frac{576966\ell-18901392}{128\ell^2+9637\ell+164296}+\frac{1861058\ell-28337858}{32\ell^2+3105\ell+38735}+\frac{-3346893\ell-33179656}{512\ell^2+21113\ell+208616}\right.\\
&\qquad\left.+\frac{640174922-247177757\ell}{512\ell^2+75117\ell+496726}+\frac{149632638-43971876\ell}{512\ell^2+91562\ell+599789}+\frac{-8189640\ell-1827686304}{512\ell^2+150141\ell+7149852}\right),
\end{aligned}$$

which we need to prove nonnegative. The sum of 6 rational expressions on the last two lines above are monotonic in ℓ (for $\ell\geq 27+\frac{1}{50}$). The terms on the first line each go to ∞ with ℓ, driving the whole expression slowly up. The whole expression has a minimum of about 0.0007 near $\ell\approx 750$. Splitting the domain for ℓ into 124 pieces, we arrive at a proof of the Main Theorem.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, BC, CANADA V6T 1Z2

Email address: bennett@math.ubc.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, BC, CANADA V6T 1Z2

Email address: gerg@math.ubc.ca

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, COLLEGE OF STATEN ISLAND AND THE GRADUATE CENTER, 2800 VICTORY BOULEVARD, STATEN ISLAND, NY, USA 10314

Email address: truculentmath@icloud.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, ROOM 121, 1984 MATHEMATICS ROAD, VANCOUVER, BC, CANADA V6T 1Z2

Email address: andrewr@math.ubc.ca