The Main Bound Computations ($5.98 \le L \le 28$)

Here we assemble bounds on the relevant pieces and combine them to accomplish the bound in the Main Theorem.

Definitions

In[1]:=

```
<< IntervalTools`
```

In[2]:=

```
{Lmin, Lmax} = {6 - 1/50, 28};
Linterval = Interval[{Lmin, Lmax}];
η[L_] = 18/(10 + 9 L);
c[L_] = 1 + 505/(111 L + 430);
r[L_] = 149/140 + 747/(283 + 36 L);
σ1[L_] = c[L] + (c[L] - 1/2)^2/r[L];
δ[L_] = 2 c[L] - σ1[L] - 1/2;
θ-η[L_] = ArcCos[(-η[L] - c[L])/r[L]];
θ1+η[L_] = ArcCos[(1 + η[L] - c[L])/r[L]];
θ1-c[L_] = ArcCos[(1 - 2 c[L])/r[L]];
θ-1/2[L_] = ArcCos[(-1/2 - c[L])/r[L]];
θ-3/2[L_] = π;
```

In[14]:=
```
Reduce[D[η[L], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[c[L], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[r[L], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[σ1[L], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[δ[L], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
```

Out[14]= $\frac{299}{50} \le L \le 28$

Out[15]= $\frac{299}{50} \le L \le 28$

Out[16]= $\frac{299}{50} \le L \le 28$

Out[17]= $\frac{299}{50} \le L \le 28$

Out[18]= $\frac{299}{50} \le L \le 28$

In[19]:=
```
η[Lint_Interval] := Block[{L}, MonotoneEnclosure[η[L], L, Lint]];
c[Lint_Interval] := Block[{L}, MonotoneEnclosure[c[L], L, Lint]];
r[Lint_Interval] := Block[{L}, MonotoneEnclosure[r[L], L, Lint]];
σ1[Lint_Interval] := Block[{L}, MonotoneEnclosure[σ1[L], L, Lint]];
δ[Lint_Interval] := Block[{L}, MonotoneEnclosure[δ[L], L, Lint]];
```

In[24]:=
```
Reduce[D[θ-η[L], L] ≥ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[θ1+η[L], L] ≤  0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[θ1-c[L], L] ≥ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
Reduce[D[θ-1/2[L], L] ≥ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
```

Out[24]= $8.18\ldots \le L \le 28$

Out[25]= $6.18\ldots \le L \le 28$

Out[26]= $\frac{-17\,617\,402 + \sqrt{197\,524\,318\,907\,258\,238}}{45\,811\,476} \le L \le 28$

Out[27]= $\frac{299}{50} \le L \le 28$

Thus, the following interval enclosures are justified. Actually, η, c, r only contain one occurrence of L, and so are automatically sharp.

In[28]:=
```
θ_{-η}[Lint_Interval] :=
  Block[{L}, PiecewiseMonotoneEnclosure[θ_{-η}[L], L, {8.18…}, Lint]];
θ_{1+η}[Lint_Interval] := Block[{L},
   PiecewiseMonotoneEnclosure[θ_{1+η}[L], L, {6.18…}, Lint]];
θ_{1-c}[Lint_Interval] := Block[{L}, PiecewiseMonotoneEnclosure[
    θ_{1-c}[L], L, {(-17 617 402 + Sqrt[197 524 318 907 258 238])/45 811 476}, Lint]];
θ_{-1/2}[Lint_Interval] := Block[{L}, MonotoneEnclosure[θ_{-1/2}[L], L, Lint]]
```

Additionally we will frequently encounter $\log\left(\frac{r}{c-1/2}\right)$, which is monotone decreasing:

In[32]:=
```
Reduce[D[Log[r[L]/(c[L] - 1/2)], L] ≤ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
```

Out[32]= $\frac{-17\,617\,402+\sqrt{197\,524\,318\,907\,258\,238}}{45\,811\,476} \le L \le 28$

In[33]:=
```
logrc[L_] = Log[r[L]/(c[L] - 1/2)];
logrc[Lint_Interval] := Block[{L}, PiecewiseMonotoneEnclosure[
    Log[r[L]/(c[L] - 1/2)], L, {(-17 617 402 + Sqrt[197 524 318 907 258 238])/45 811 476}, Lint]]
```

In[35]:=
```
Er[a_, d_, T_] =
  (4 (4 + 3 π))/(45 ((17 + 2 a)^2 + 4 T^2)^(3/2)) - (4 T)/(3 ((17 + 2 a)^2 + 4 T^2)) + (8 + 6 π)/(45 ((17 + 2 a - 2 d)^2 + 4 T^2)^(3/2)) +
   (2 T)/(3 ((17 + 2 a - 2 d)^2 + 4 T^2)) + (8 + 6 π)/(45 ((17 + 2 a + 2 d)^2 + 4 T^2)^(3/2)) + (2 T)/(3 ((17 + 2 a + 2 d)^2 + 4 T^2)) +
   2 ArcTan[(1 + 2 a)/(2 T)] + 2 ArcTan[(5 + 2 a)/(2 T)] + 2 ArcTan[(9 + 2 a)/(2 T)] + 2 ArcTan[(13 + 2 a)/(2 T)] -
   1/2 (15 + 2 a) ArcTan[(17 + 2 a)/(2 T)] - ArcTan[(1 + 2 a - 2 d)/(2 T)] - ArcTan[(5 + 2 a - 2 d)/(2 T)] -
   ArcTan[(9 + 2 a - 2 d)/(2 T)] - ArcTan[(13 + 2 a - 2 d)/(2 T)] + 1/4 (15 + 2 a - 2 d) ArcTan[(17 + 2 a - 2 d)/(2 T)] -
   ArcTan[(1 + 2 a + 2 d)/(2 T)] - ArcTan[(5 + 2 a + 2 d)/(2 T)] - ArcTan[(9 + 2 a + 2 d)/(2 T)] -
   ArcTan[(13 + 2 a + 2 d)/(2 T)] + 1/4 (15 + 2 a + 2 d) ArcTan[(17 + 2 a + 2 d)/(2 T)] +
   1/2 T Log[1 + (17 + 2 a)^2/(4 T^2)] - 1/4 T Log[1 + (17 + 2 a - 2 d)^2/(4 T^2)] - 1/4 T Log[1 + (17 + 2 a + 2 d)^2/(4 T^2)];
```

Bounds on the pieces

Domains for L, η, c, r, σ_1, δ

In[36]:=
```
ηinterval = η[Linterval]
cinterval = c[Linterval]
rinterval = r[Linterval]
N[{ηinterval, cinterval, rinterval}, 3]
```

Out[36]= Interval$\left[\left\{\frac{9}{131}, \frac{900}{3191}\right\}\right]$

Out[37]= Interval$\left[\left\{\frac{4043}{3538}, \frac{79\,939}{54\,689}\right\}\right]$

Out[38]= Interval$\left[\left\{\frac{296\,939}{180\,740}, \frac{4\,470\,593}{1\,743\,980}\right\}\right]$

Out[39]= {Interval[{0.0685, 0.283}], Interval[{1.14, 1.46}], Interval[{1.64, 2.57}]}

In[40]:=
```
σ1interval = σ1[Linterval]
δinterval = δ[Linterval]
N[{σ1interval, δinterval}, 3]
```

Out[40]= Interval$\left[\left\{\frac{2\,591\,037\,761\,033}{1\,858\,458\,651\,958}, \frac{24\,368\,631\,908\,992\,198}{13\,371\,037\,238\,695\,553}\right\}\right]$

Out[41]= Interval$\left[\left\{\frac{363\,594\,079\,407}{929\,229\,325\,979}, \frac{16\,069\,566\,216\,379\,263}{26\,742\,074\,477\,391\,106}\right\}\right]$

Out[42]= {Interval[{1.39, 1.82}], Interval[{0.391, 0.602}]}

In[43]:=
```
θmeinterval = θ_{-η}[Linterval];
θ1peinterval = θ_{1+η}[Linterval];
θ1mcinterval = θ_{1-c}[Linterval];
θm12interval = θ_{-1/2}[Linterval];
N[{θmeinterval, θ1peinterval, θ1mcinterval, θm12interval}, 3]
```

Out[47]= {Interval[{2.31, 2.40}], Interval[{1.61, 1.64}],
Interval[{2.40, 2.47}], Interval[{2.44, 3.13}]}

Inequalities:

$-\frac{3}{2} < c[L] - r[L] < \frac{-1}{2} < 1 - c[L] \le -\eta[L] < 0 < 1 < 1 + \eta[L] < c[L] < \sigma_1[L] < c[L] + r[L]$

In[48]:= $-\frac{3}{2} < c[L] - r[L] < \frac{-1}{2} < 1 - c[L] < -\eta[L] < 0 < 1 < 1 + \eta[L] < c[L] < \sigma_1[L] < c[L] + r[L]$

Out[48]= $-\frac{3}{2} < -\frac{9}{140} - \frac{747}{283 + 36\,L} + \frac{505}{430 + 111\,L} < -\frac{1}{2} < -\frac{505}{430 + 111\,L} < -\frac{18}{10 + 9\,L} < 0 < 1 < 1 + \frac{18}{10 + 9\,L} < 1 + \frac{505}{430 + 111\,L} < 1 + \frac{505}{430 + 111\,L} + \frac{\left(\frac{1}{2} + \frac{505}{430+111\,L}\right)^2}{\frac{149}{140} + \frac{747}{283+36\,L}} < \frac{289}{140} + \frac{747}{283 + 36\,L} + \frac{505}{430 + 111\,L}$

In[49]:= `N[Reduce[`$-\frac{3}{2} < c[L] - r[L] < \frac{-1}{2} < 1 - c[L] < -\eta[L] < 0 < 1 < 1 + \eta[L] < c[L] < \sigma_1[L] < c[L] + r[L]$`, L], 30]`

`Reduce[`$-\frac{3}{2} < c[L] - r[L] < \frac{-1}{2} < 1 - c[L] < -\eta[L] < 0 < 1 < 1 + \eta[L] < c[L] < \sigma_1[L] < c[L] + r[L]$` && Lmin ≤ L ≤ Lmax, L]`

Out[49]= $5.22522522522522522522522522523 < L < 28.0146690374108849373805037045$

Out[50]= $\frac{299}{50} \le L \le 28$

In[51]:= `θ1peinterval ≤ 2.1`

Out[51]= `True`

In[52]:= `Reduce[r[L] ≥ 2 c[L] - 1 && Lmin ≤ L ≤ Lmax]`

Out[52]= $\frac{299}{50} \le L \le 28$

Bound: $\left| g(a, T) \right| \le \frac{2-a}{50\,T}$

In the file “ZerosOfLFunctions-Prop32.nb”.

Bound: $\frac{E_\delta}{\pi} \le \frac{(640+216\,a)\,d-112-39\,a}{1536\,(3\,T+3\,a-1)} + \frac{1}{2^{10}}$

This is in the file ZerosOfLFunctions-E.nb.

Bound: $\left| g(a, T) \right| + \frac{E_\delta}{\pi} \le \frac{1}{2^{10}} + \frac{6/55}{T-1/5}$

In[53]:= `bound[1] = `$\frac{1}{2^{10}} + \frac{6/55}{T - 1/5}$`;`

In[54]:=
```
Clear[f];
f[a_, d_, T_] = bound[1] - ((2 - a)/(50 T) + ((640 + 216 a) d - 112 - 39 a)/(1536 (3 T + 3 a - 1)) + 1/2^10);
(* for each a, the function f is monotone in d, and rational in T *)
f[a_, dint_Interval, Tint_Interval] :=
  IntervalHull[f[a, Min[dint], Tint], f[a, Max[dint], Tint]];
f[a_, d_, Tint_Interval] := Module[{T, tmin = Min[Tint], tmax = Max[Tint], min, max},
   (* at this point,
   a and d are specific numbers, and what's left is rational in T *)
   min = First[Minimize[{f[a, d, T], tmin ≤ T ≤ tmax}, T, Reals]];
   max = First[Maximize[{f[a, d, T], tmin ≤ T ≤ tmax}, T, Reals]];
   Interval[{min, max}]];
```

In[58]:=
```
N[δinterval, 4]
```

Out[58]=
```
Interval[{0.3912, 0.6010}]
```

In[59]:=
```
f[0, δinterval, Interval[{5 / 7, ∞}]]
f[1, δinterval, Interval[{5 / 7, ∞}]]
```

Minimize: The minimum is not attained at any point satisfying the given constraints.

Minimize: The minimum is not attained at any point satisfying the given constraints.

Out[59]= $\text{Interval}\left[\left\{0, \frac{25\,273\,189\,621\,447\,327}{327\,088\,722\,744\,608\,000}\right\}\right]$

Minimize: The minimum is not attained at any point satisfying the given constraints.

Minimize: The minimum is not attained at any point satisfying the given constraints.

Out[60]= $\text{Interval}\left[\left\{0, \frac{2\,944\,612\,732\,439\,297\,983}{18\,971\,145\,919\,187\,264\,000}\right\}\right]$

These warnings are not worrisome: they merely acknowledge that the extreme values happen as as $T \to \infty$.

Bound: $\frac{2}{\pi}\,\text{Log}[\text{Zeta}[\sigma_1]] - \frac{\text{Log}[\text{Zeta}[2\,c]]}{\text{Log}\left[\frac{r}{c-1/2}\right]} \le \frac{16\,489+14\,928\,L+402\,L^2}{380\,433+35\,739\,L+1024\,L^2}$

In[61]:=
```
bound[2] = (16489 + 14928 L + 402 L^2)/(380433 + 35739 L + 1024 L^2);
```

In[62]:=
```
Clear[f];
f[L_] = bound[2] - (2/π Log[Zeta[σ1[L]]] - Log[Zeta[2 c[L]]]/Log[r[L]/(c[L] - 1/2)]);
```

In[64]:=
```
Reduce[D[bound[2]] ≥ 0 && Lmin ≤ L ≤ Lmax, L, Reals]
```

Out[64]= $\frac{299}{50} \le L \le 28$

In[65]:=
$$\texttt{f[Lint\_Interval] := Module}\Big[\{\texttt{rat, rat2, rat3, L}\},\ \texttt{rat = MonotoneEnclosure}\Big[\frac{16\,489+14\,928\,L+402\,L^2}{380\,433+35\,739\,L+1024\,L^2},\texttt{L, Lint}\Big];\ \texttt{rat} - \left(\frac{2}{\pi}\,\texttt{Log[Zeta[}\sigma_1\texttt{[Lint]]]} - \frac{\texttt{Log[Zeta[2 c[Lint]]]}}{\texttt{logrc[Lint]}}\right)\Big];$$

In[66]:= `N[f[Interval[{10, 11}]], 30]`

Out[66]= `Interval[{-0.0212058570731966894511238579019, 0.0220622259592663757862354551166}]`

In[67]:=
```
Timing[{t0, t1, t2} = ProveNonNegative[f, f, {Linterval}, MaxDepth → 20];]
Length /@ {t0, t1, t2}
```

Out[67]= `{5.35056, Null}`

Out[68]= `{0, 0, 1139}`

Bound: $\frac{1/\pi}{\text{Log}[r/(c-1/2)]}\,\kappa_1 L \le \frac{62\,044+305\,799\,L+18\,763\,L^2+235\,L^3}{2\,(243\,814+38\,507\,L+512\,L^2)}$

In[69]:=
$$\texttt{bound[3]} = \frac{62\,044+305\,799\,L+18\,763\,L^2+235\,L^3}{2\,(243\,814+38\,507\,L+512\,L^2)};$$

In[70]:=
$$\kappa_1\texttt{[L\_]} = (\theta_{-\eta}\texttt{[L]} - \theta_{1+\eta}\texttt{[L]}) * \frac{1+\eta\texttt{[L]} - \texttt{c[L]}}{2} - (\pi - \theta_{-\eta}\texttt{[L]})\left(\texttt{c[L]} - \frac{1}{2}\right) + \frac{\texttt{r[L]}\,(\texttt{Sin[}\theta_{-\eta}\texttt{[L]]} + \texttt{Sin[}\theta_{1+\eta}\texttt{[L]]})}{2};$$

In[71]:=
$$\texttt{D[}\theta_{-\eta}\texttt{[L]} - \theta_{1+\eta}\texttt{[L], L]}$$
$$\texttt{Reduce[D[}\theta_{-\eta}\texttt{[L]} - \theta_{1+\eta}\texttt{[L], L]} == 0\ \&\&\ \texttt{Lmin} \le \texttt{L} \le \texttt{Lmax]}$$

Out[71]=
$$-\frac{\dfrac{\frac{162}{(10+9L)^2}+\frac{56\,055}{(430+111L)^2}}{\frac{149}{140}+\frac{747}{283+36L}} + \dfrac{26\,892\left(-1-\frac{18}{10+9L}-\frac{505}{430+111L}\right)}{(283+36L)^2\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}{\sqrt{1-\dfrac{\left(-1-\frac{18}{10+9L}-\frac{505}{430+111L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}} + \frac{\dfrac{-\frac{162}{(10+9L)^2}+\frac{56\,055}{(430+111L)^2}}{\frac{149}{140}+\frac{747}{283+36L}} + \dfrac{26\,892\left(\frac{18}{10+9L}-\frac{505}{430+111L}\right)}{(283+36L)^2\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}{\sqrt{1-\dfrac{\left(\frac{18}{10+9L}-\frac{505}{430+111L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}}$$

Out[72]= `L == 7.73...`

In[73]:=
$$\texttt{D}\left[\frac{1+\eta\texttt{[L]}-\texttt{c[L]}}{2},\texttt{L}\right]$$
$$\texttt{Reduce}\left[\texttt{D}\left[\frac{1+\eta\texttt{[L]}-\texttt{c[L]}}{2},\texttt{L}\right] \ge 0\ \&\&\ \texttt{Lmin} \le \texttt{L} \le \texttt{Lmax}\right]$$

Out[73]=
$$\frac{1}{2}\left(-\frac{162}{(10+9L)^2}+\frac{56\,055}{(430+111L)^2}\right)$$

Out[74]=
$$\frac{299}{50} \le L \le 28$$

In[75]:=

```
D[(1 + η[L] - c[L])/2, L]
Reduce[D[(1 + η[L] - c[L])/2, L] ≥ 0 && Lmin ≤ L ≤ Lmax]
```

Out[75]= $$\frac{1}{2}\left(-\frac{162}{(10+9L)^2}+\frac{56055}{(430+111L)^2}\right)$$

Out[76]= $$\frac{299}{50}\leq L\leq 28$$

In[77]:=

```
temp = Simplify[(r[L] Sin[θ_{-η}[L]])]
D[temp, L]
Reduce[D[temp, L] ≤ 0 && Lmin ≤ L ≤ Lmax]
```

Out[77]= $$\left(\frac{149}{140}+\frac{747}{283+36L}\right)\sqrt{1-\frac{\left(1+\frac{18}{10+9L}+\frac{505}{430+111L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}$$

Out[78]= $$\frac{\left(\frac{149}{140}+\frac{747}{283+36L}\right)\left(-\frac{2\left(-\frac{162}{(10+9L)^2}-\frac{56055}{(430+111L)^2}\right)\left(1+\frac{18}{10+9L}+\frac{505}{430+111L}\right)}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}-\frac{53784\left(1+\frac{18}{10+9L}+\frac{505}{430+111L}\right)^2}{(283+36L)^2\left(\frac{149}{140}+\frac{747}{283+36L}\right)^3}\right)}{2\sqrt{1-\frac{\left(1+\frac{18}{10+9L}+\frac{505}{430+111L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}}-\frac{26892\sqrt{1-\frac{\left(1+\frac{18}{10+9L}+\frac{505}{430+111L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36L}\right)^2}}}{(283+36L)^2}$$

Out[79]= $$\frac{299}{50}\leq L\leq 28$$

In[80]:=
```
temp = Simplify[(r[L] Sin[θ_{1+η}[L]])]
D[temp, L]
Reduce[D[temp, L] ≤ 0 && Lmin ≤ L ≤ Lmax]
```

Out[80]=
$$\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)\sqrt{1-\frac{\left(\frac{18}{10+9\,L}-\frac{505}{430+111\,L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)^2}}$$

Out[81]=
$$\frac{\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)\left(-\frac{2\left(-\frac{162}{(10+9\,L)^2}+\frac{56\,055}{(430+111\,L)^2}\right)\left(\frac{18}{10+9\,L}-\frac{505}{430+111\,L}\right)}{\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)^2}-\frac{53\,784\left(\frac{18}{10+9\,L}-\frac{505}{430+111\,L}\right)^2}{(283+36\,L)^2\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)^3}\right)}{2\sqrt{1-\frac{\left(\frac{18}{10+9\,L}-\frac{505}{430+111\,L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)^2}}}-\frac{26\,892\sqrt{1-\frac{\left(\frac{18}{10+9\,L}-\frac{505}{430+111\,L}\right)^2}{\left(\frac{149}{140}+\frac{747}{283+36\,L}\right)^2}}}{(283+36\,L)^2}$$

Out[82]=
$$\frac{299}{50}\le L\le 28$$

In[83]:=
```
κ_1[Lint_Interval] := Module[{L, term1, term2, term3, term4},
    term1 = PiecewiseMonotoneEnclosure[θ_{-η}[L] - θ_{1+η}[L], L, {7.73...}, Lint];
    term2 = MonotoneEnclosure[(1 + η[L] - c[L])/2, L, Lint];
    term3 = MonotoneEnclosure[r[L] Sin[θ_{-η}[L]], L, Lint];
    term4 = MonotoneEnclosure[r[L] Sin[θ_{1+η}[L]], L, Lint];

    term1 * term2 - (π - θ_{-η}[Lint]) (c[Lint] - 1/2) + (term3 + term4)/2];
```

In[84]:=
```
Clear[f];
f[L_] = bound[3] - (1/π)/logrc[L] κ_1[L];
```

In[86]:=
```
Reduce[D[bound[3], L] ≥ 0 && Lmin ≤ L ≤ Lmax]
```

Out[86]=
$$\frac{299}{50}\le L\le 28$$

In[87]:=
```
f[Lint_Interval] := Module[{L, rat},
    rat = MonotoneEnclosure[(62 044 + 305 799 L + 18 763 L^2 + 235 L^3)/(2 (243 814 + 38 507 L + 512 L^2)), L, Lint];
    rat - (1/π)/logrc[Lint] κ_1[Lint] * Lint];
```

In[88]:=
```
AbsoluteTiming[
  {fail, undecided, proven} = ProveNonNegative[f, f, {Linterval}, MaxDepth → 20];]
{fail, undecided, Length[proven]}
```

Out[88]= {168.689, Null}

Out[89]= {{}, {}, 12 937}

Bound:

$$\frac{\mathrm{Log[Zeta[c]]}}{2\,\mathrm{Log}\left[\frac{r}{c-1/2}\right]}\left(\frac{3}{2}+\frac{1}{2J_1}+\frac{1}{J_2}+\frac{\theta_{1+\eta}}{\pi}+\left(1-\frac{1}{J_2}\right)\frac{\theta_{1-c}}{\pi}-\frac{\theta_{-\eta}}{\pi}\right)\leq\frac{199\,847+79\,917\,L+5034\,L^2+7\,L^3}{2\left(192\,512+31\,343\,L+512\,L^2\right)}$$

In[90]:=
```
bound[4] = (199 847 + 79 917 L + 5034 L^2 + 7 L^3) / (2 (192 512 + 31 343 L + 512 L^2));
```

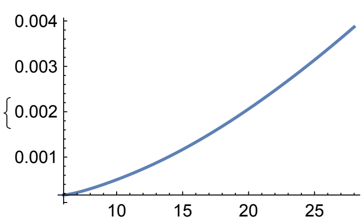
In[91]:=
```
Clear[f];
f[L_] = Block[{J1 = 64, J2 = 24},
  bound[4] - Log[Zeta[c[L]]] / (2 logrc[L]) (3/2 + 1/(2 J1) + 1/J2 + θ_{1+η}[L]/π + (1 - 1/J2) θ_{1-c}[L]/π - θ_{-η}[L]/π)]
```

Out[92]=

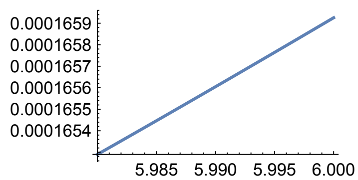
$$\frac{199\,847+79\,917\,L+5034\,L^2+7\,L^3}{2\left(192\,512+31\,343\,L+512\,L^2\right)}-\frac{1}{2\,\mathrm{Log}\left[\frac{\frac{149}{140}+\frac{747}{283+36\,L}}{\frac{1}{2}+\frac{505}{430+111\,L}}\right]}\left(\frac{595}{384}-\frac{\mathrm{ArcCos}\left[\frac{-1-\frac{18}{10+9\,L}-\frac{505}{430+111\,L}}{\frac{149}{140}+\frac{747}{283+36\,L}}\right]}{\pi}+\frac{\mathrm{ArcCos}\left[\frac{\frac{18}{10+9\,L}-\frac{505}{430+111\,L}}{\frac{149}{140}+\frac{747}{283+36\,L}}\right]}{\pi}+\frac{23\,\mathrm{ArcCos}\left[\frac{1-2\left(1+\frac{505}{430+111\,L}\right)}{\frac{149}{140}+\frac{747}{283+36\,L}}\right]}{24\,\pi}\right)\mathrm{Log}\left[\mathrm{Zeta}\left[1+\frac{505}{430+111\,L}\right]\right]$$

In[93]:=
```
{Plot[f[L], {L, Lmin, Lmax}, PlotRange → All],
 Plot[f[L], {L, Lmin, 6}, PlotRange → All]}
```

Out[93]= {



,



}

In[94]:=
```
Solve[D[bound[4], L] == 0, L, Reals]
```

Out[94]= {}

In[95]:=

$$\texttt{f[Lint\_Interval] := Module}\Big[\{\texttt{L, bound, J1 = 64, J2 = 24}\},$$
$$\texttt{bound = MonotoneEnclosure}\left[\frac{199\,847+79\,917\,L+5034\,L^2+7\,L^3}{2\left(192\,512+31\,343\,L+512\,L^2\right)},\ L,\ \texttt{Lint}\right];$$
$$\texttt{bound} - \frac{\texttt{Log[Zeta[}c\texttt{[Lint]]]}}{2\,\texttt{logrc[Lint]}}\left(\frac{3}{2}+\frac{1}{2\,\texttt{J1}}+\frac{1}{\texttt{J2}}+\frac{\theta_{1+\eta}\texttt{[Lint]}}{\pi}+\left(1-\frac{1}{\texttt{J2}}\right)\frac{\theta_{1-c}\texttt{[Lint]}}{\pi}-\frac{\theta_{-\eta}\texttt{[Lint]}}{\pi}\right)\Big];$$

In[96]:=

```
AbsoluteTiming[
 {fail, undecided, proven} = ProveNonNegative[f, f, {Linterval}, MaxDepth → 20];]
{fail, undecided, Length[proven]}
```

Out[96]= {15.3762, Null}

Out[97]= {{}, {}, 2426}

Bound: $\frac{\text{Log[Zeta}[1+\eta]]}{2\,\text{Log}\left[\frac{r}{c-1/2}\right]}\left(-\frac{\theta_{1+\eta}}{\pi}+\frac{\theta_{1-c}}{\pi}+\frac{\theta_{-\eta}}{\pi}-\frac{1}{2}\right)\leq\frac{-82\,709+17\,494\,L+3466\,L^2-2\,L^3}{2\left(-188\,287+62\,894\,L+512\,L^2\right)}$

In[98]:=

$$\texttt{bound[5]} = \frac{-82\,709+17\,494\,L+3466\,L^2-2\,L^3}{2\left(-188\,287+62\,894\,L+512\,L^2\right)};$$

In[99]:= `Clear[f];`

$$\texttt{f[L\_] = bound[5]} - \frac{\texttt{Log[Zeta[}1+\eta\texttt{[}L\texttt{]]]}}{2\,\texttt{logrc[}L\texttt{]}}\left(-\frac{\theta_{1+\eta}[L]}{\pi}+\frac{\theta_{1-c}[L]}{\pi}+\frac{\theta_{-\eta}[L]}{\pi}-\frac{1}{2}\right)$$

Out[100]=

$$\frac{-82\,709+17\,494\,L+3466\,L^2-2\,L^3}{2\left(-188\,287+62\,894\,L+512\,L^2\right)} - \frac{\left(-\frac{1}{2}+\frac{\text{ArcCos}\left[\frac{-1-\frac{18}{10+9L}-\frac{505}{430+111L}}{\frac{149}{140}+\frac{747}{283+36L}}\right]}{\pi}-\frac{\text{ArcCos}\left[\frac{\frac{18}{10+9L}-\frac{505}{430+111L}}{\frac{149}{140}+\frac{747}{283+36L}}\right]}{\pi}+\frac{\text{ArcCos}\left[\frac{1-2\left(1+\frac{505}{430+111L}\right)}{\frac{149}{140}+\frac{747}{283+36L}}\right]}{\pi}\right)\text{Log}\left[\text{Zeta}\left[1+\frac{18}{10+9L}\right]\right]}{2\,\text{Log}\left[\frac{\frac{149}{140}+\frac{747}{283+36L}}{\frac{1}{2}+\frac{505}{430+111L}}\right]}$$

In[101]:= `Reduce[D[bound[5], L] ≥ 0 && Lmin ≤ L ≤ Lmax]`

Out[101]= $\frac{299}{50} \leq L \leq 28$

In[102]:=

$$\texttt{f[Lint\_Interval] := Module}\Big[\{\texttt{L, rat, term}\},$$
$$\texttt{rat = MonotoneEnclosure}\left[\frac{-82\,709+17\,494\,L+3466\,L^2-2\,L^3}{2\left(-188\,287+62\,894\,L+512\,L^2\right)},\ L,\ \texttt{Lint}\right];$$
$$\texttt{term} = \frac{\texttt{Log[Zeta[}1+\eta\texttt{[Lint]]]}}{2\,\texttt{logrc[Lint]}}\left(-\frac{\theta_{1+\eta}\texttt{[Lint]}}{\pi}+\frac{\theta_{1-c}\texttt{[Lint]}}{\pi}+\frac{\theta_{-\eta}\texttt{[Lint]}}{\pi}-\frac{1}{2}\right);$$
$$\texttt{rat - term}\Big]$$

In[103]:=
```
AbsoluteTiming[
 {fail, undecided, proven} = ProveNonNegative[f, f, {Linterval}, MaxDepth → 20];]
{fail, undecided, Length[proven]}
```

Out[103]= {2.5834, Null}

Out[104]= {{}, {}, 428}

Bound: $\kappa_2 = \frac{\pi}{4 J_1}\left(\text{Log}[\text{Zeta}[c+r]] + 2\,\text{Sum}\left[\text{Log}\left[\text{Zeta}\left[c + r\,\text{Cos}\left[\frac{\pi j}{2 J_1}\right]\right]\right], \{j, 1, J_1 - 1\}\right]\right)$

$$\frac{\kappa_2/\pi}{\text{Log}[r/(c-1/2)]} \le \frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2}$$

In[105]:= bound[6] = $\frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2}$;

In[106]:= Clear[f];

f[L_] = bound[6] - Block[{J1 = 64}, $\frac{1/\pi}{\text{logrc}[L]}\,\frac{\pi}{4\,J1}$
  (Log[Zeta[c[L] + r[L]]] + 2 Sum[Log[Zeta[c[L] + r[L] Cos[$\frac{\pi\, j}{2\, J1}$]]], {j, 1, J1 - 1}])];

In[108]:= Reduce[D[$\frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2}$, L] ≥ 0 && Lmin ≤ L ≤ Lmax]

Out[108]= $\frac{299}{50} \le L \le 28$

In[109]:= f[Lint_Interval] := Module[{L, rat, J1 = 64},
  rat = MonotoneEnclosure[$\frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2}$, L, Lint];
  rat - $\frac{1/\pi}{\text{logrc}[\text{Lint}]}\,\frac{\pi}{4\,J1}$ (Log[Zeta[c[Lint] + r[Lint]]] +
    2 Sum[Log[Zeta[c[Lint] + r[Lint] Cos[$\frac{\pi\, j}{2\, J1}$]]], {j, 1, J1 - 1}])];

In[110]:=
```
AbsoluteTiming[
 {fail, undecided, proven} = ProveNonNegative[f, f, {Linterval}, MaxDepth → 20];]
{fail, undecided, Length[proven]}
```

Out[110]= {52.016, Null}

Out[111]= {{}, {}, 241}

Bound: $\kappa_3 = \frac{\pi-\theta_{1-c}}{2 J_2}\left(\text{Log}[\text{Zeta}[1-c+r]] + 2\,\text{Sum}\left[\text{Log}\left[\text{Zeta}\left[1-c-r\,\text{Cos}\left[\theta_{1-c}+(\pi-\theta_{1-c})\frac{j}{J_2}\right]\right]\right], \{j, 1, J_2-1\}\right]\right)$

$$\frac{\kappa_3/\pi}{\text{Log}[r/(c-1/2)]} \leq \frac{-6171+827 L+537 L^2}{-92\,820+31\,635 L+1024 L^2}$$

```
In[112]:= bound[7] = (-6171 + 827 L + 537 L^2)/(-92 820 + 31 635 L + 1024 L^2);

In[113]:= Clear[f];
f[L_] = bound[7] - Block[{J2 = 24}, (1/π)/logrc[L] (π - θ1-c[L])/(2 J2) (Log[Zeta[1 - c[L] + r[L]]] +
      2 Sum[Log[Zeta[1 - c[L] - r[L] Cos[θ1-c[L] (1 - j/J2) + π j/J2]]], {j, 1, J2 - 1}])];

In[115]:= Reduce[D[bound[7], L] ≥ 0 && L ≥ 27]
Reduce[D[1 - c[L] + r[L], L] ≤ 0 && L ≥ 27]

Out[115]= L ≥ 27

Out[116]= L ≥ 27

In[117]:= f[Lint_] := Module[{L, rat, J2, rmcp1, insidezeta},
   J2 = 24;
   rat = MonotoneEnclosure[(-6171 + 827 L + 537 L^2)/(-92 820 + 31 635 L + 1024 L^2), L, Lint];
   rmcp1 = Log[Zeta[MonotoneEnclosure[1 - c[L] + r[L], L, Lint]]];
   insidezeta[j_] := IntervalIntersection[Interval[{Min[c[Lint]], ∞}],
     1 - c[Lint] - r[Lint] * Cos[θ1-c[Lint] * (1 - j/J2) + π j/J2]];
   rat - (1/π)/logrc[Lint] (π - θ1-c[Lint])/(2 J2)
     (rmcp1 + 2 Sum[Log[Zeta[insidezeta[j]]], {j, 1, J2 - 1}])];

In[118]:= N[f[Interval[{6, 7}]], 30]
N[f[Interval[{27, 28}]], 30]

Out[118]= Interval[{-0.0178224753631433564037923040419, 0.0177085739619176247288230878377}]

Out[119]= Interval[{-0.00424947728654049879760181439754, 0.0131160377914591198610678968533}]

In[120]:= Timing[Length /@ ProveNonNegative[f, f, {Linterval}, MaxDepth → 25]]

Out[120]= {42.7001, {0, 0, 239}}
```

Bound: $\frac{\kappa_4/\pi}{\text{Log}[r/(c-1/2)]} \leq \frac{110\,537-2290 L-333 L^2+2 L^3}{-49\,052+314\,027 L+1024 L^2}$

```
In[121]:= bound[8] = (110 537 - 2290 L - 333 L^2 + 2 L^3)/(-49 052 + 314 027 L + 1024 L^2);
```

The quantity κ_4 is defined by

In[122]:=
```
Clear[K4];
K4[c_, r_, η_] = 1/4 Block[{σ = c + r Cos[θ], t = r Sin[θ]}, Integrate[
    (1 + η - σ) (2 t - 4 + 7/19 ((1 + σ)^2 + (t - 2)^2)), {θ, ArcCos[(1 + η - c)/r], ArcCos[(-η - c)/r]}]]
```

Out[122]=

$$\frac{1}{152}\Bigg(20\,(1-c+\eta)^2-20\,(-1+c-\eta)\,(c+\eta)-2\,r\,\big(-55+14\,c^2+7\,r^2+c\,(7-21\,\eta)-21\,\eta\big)\sqrt{-\frac{c^2-r^2+2\,c\,\eta+\eta^2}{r^2}}+2\,r\,\big(-48+14\,c^2+7\,r^2-7\,c\,(-2+\eta)-7\,\eta\big)\sqrt{1-\frac{(1-c+\eta)^2}{r^2}}+2\,\big(41+7\,c^3+c\,(-55+14\,r^2-14\,\eta)-7\,c^2\,(-1+\eta)+(41-7\,r^2)\,\eta\big)\,\mathrm{ArcCos}\Big[\frac{1-c+\eta}{r}\Big]-2\,\big(41+7\,c^3+c\,(-55+14\,r^2-14\,\eta)-7\,c^2\,(-1+\eta)+(41-7\,r^2)\,\eta\big)\,\mathrm{ArcCos}\Big[-\frac{c+\eta}{r}\Big]-5\,r^2\,\mathrm{Cos}\Big[2\,\mathrm{ArcCos}\Big[\frac{1-c+\eta}{r}\Big]\Big]+5\,r^2\,\mathrm{Cos}\Big[2\,\mathrm{ArcCos}\Big[-\frac{c+\eta}{r}\Big]\Big]\Bigg)$$

Some nontrivial rearranging work gives

In[123]:=
```
κ4[c_, r_, η_] := Block[{α1 = Sqrt[(-1 + c + r - η) (1 - c + r + η)], α2 = Sqrt[-(c - r + η) (c + r + η)]},
  1/76 (α1 (-48 + 14 c + 14 c^2 + 7 r^2 - 7 η - 7 c η) + α2 (55 - 7 c - 14 c^2 - 7 r^2 + 21 η + 21 c η) +
     5 (1 + 2 η)^2 + (41 - 55 c + 7 c^2 + 7 c^3 + 14 c r^2 + 41 η - 14 c η - 7 c^2 η - 7 r^2 η)
      (ArcTan[(-1 + c - η)/α1] - ArcTan[(c + η)/α2]))]
```

Mathematica is unable to ascertain if the two expressions are equivalent, but they are.

In[124]:=
```
FullSimplify[K4[c, r, η] - κ4[c, r, η],
 Assumptions → -3/2 < c - r < -1/2 < 1 - c < -η < 0 < 1 < 1 + η < c < c + r]
```

Out[124]=

$$-\frac{1}{152}\Big(c\,\big(-55+7\,c\,(1+c)+14\,r^2\big)-7\,\big(c\,(2+c)+r^2\big)\,\eta+41\,(1+\eta)\Big)\Bigg(\pi+2\,i\,\mathrm{ArcCosh}\Big[\frac{1-c+\eta}{r}\Big]+2\,\mathrm{ArcSin}\Big[\frac{c+\eta}{r}\Big]+2\,\mathrm{ArcTan}\Big[\frac{-1+c-\eta}{\sqrt{(-1+c+r-\eta)\,(1-c+r+\eta)}}\Big]-2\,i\,\mathrm{ArcTanh}\Big[\frac{c+\eta}{\sqrt{(c-r+\eta)\,(c+r+\eta)}}\Big]\Bigg)$$

```
MinMax[Table[Block[{k = RandomReal[{Lmin, Lmax}]},
   N[K4[c[k], r[k], η[k]] - κ4[c[k], r[k], η[k]], 20]], {10 000}]]
```

Out[126]= $\{-1.4988\times10^{-15},\ 1.41553\times10^{-15}\}$

Now we turn to developing an interval enclosure for κ_4.

```
In[128]:= (* the alphas are monotone decreasing*)
Reduce[
 Lmin ≤ L ≤ Lmax && D[(-1 + c[L] + r[L] - η[L]) (1 - c[L] + r[L] + η[L]), L] ≤ 0, L, Reals]
Reduce[Lmin ≤ L ≤ Lmax && D[-(c[L] - r[L] + η[L]) (c[L] + r[L] + η[L]), L] ≤ 0, L, Reals]
```

Out[128]= $\frac{299}{50} \leq L \leq 28$

Out[129]= $\frac{299}{50} \leq L \leq 28$

```
In[130]:= (* the alpha coefficients are monotone, one decreasing and one increasing *)
Reduce[Lmin ≤ L ≤ Lmax &&
  D[(-48 + 14 c[L] + 14 c[L]^2 + 7 r[L]^2 - 7 η[L] - 7 c[L] × η[L]), L] ≤ 0, L, Reals]
Reduce[Lmin ≤ L ≤ Lmax &&
  D[(55 - 7 c[L] - 14 c[L]^2 - 7 r[L]^2 + 21 η[L] + 21 c[L] × η[L]), L] ≥ 0, L, Reals]
```

Out[130]= $\frac{299}{50} \leq L \leq 28$

Out[131]= $\frac{299}{50} \leq L \leq 28$

```
In[132]:= (*the arctan coefficient is monotone decreasing *)
Reduce[Lmin ≤ L ≤ Lmax &&
  D[(41 - 55 c[L] + 7 c[L]^2 + 7 c[L]^3 + 14 c[L] r[L]^2 + 41 η[L] - 14 c[L] × η[L] -
      7 c[L]^2 η[L] - 7 r[L]^2 η[L]), L] ≤ 0, L, Reals]
```

Out[132]= $\frac{299}{50} \leq L \leq 28$

```
In[133]:= kappa4[L_] = κ4[c[L], r[L], η[L]];
kappa4[Lint_Interval] := Module[{a1c, a2c, arctanc, L, α1, α2},
  α1 =
   Sqrt[MonotoneEnclosure[(-1 + c[L] + r[L] - η[L]) (1 - c[L] + r[L] + η[L]), L, Lint]];
  α2 = Sqrt[MonotoneEnclosure[-(c[L] - r[L] + η[L]) (c[L] + r[L] + η[L]), L, Lint]];
  a1c = MonotoneEnclosure[
    (-48 + 14 c[L] + 14 c[L]^2 + 7 r[L]^2 - 7 η[L] - 7 c[L] × η[L]), L, Lint];
  a2c = MonotoneEnclosure[(55 - 7 c[L] - 14 c[L]^2 - 7 r[L]^2 + 21 η[L] + 21 c[L] × η[L]),
    L, Lint];
  arctanc = MonotoneEnclosure[(41 - 55 c[L] + 7 c[L]^2 + 7 c[L]^3 + 14 c[L] r[L]^2 +
      41 η[L] - 14 c[L] × η[L] - 7 c[L]^2 η[L] - 7 r[L]^2 η[L]), L, Lint];
```

$$\frac{1}{76}\left(\alpha1 * a1c + \alpha2 * a2c + 5\,(1 + 2\,\eta[Lint])^2 + arctanc\left(ArcTan\left[\frac{-1 + c[Lint] - \eta[Lint]}{\alpha1}\right] - ArcTan\left[\frac{c[Lint] + \eta[Lint]}{\alpha2}\right]\right)\right)]$$

In[135]:= Reduce[Lmin ≤ L ≤ Lmax && D[(110 537 - 2290 L - 333 L^2 + 2 L^3)/(-49 052 + 314 027 L + 1024 L^2), L] ≤ 0, L, Reals]

Out[135]= $\frac{299}{50} \leq L \leq 28$

```
In[136]:= Clear[f];
f[L_] = bound[8] - (kappa4[L] / π)/logrc[L];
f[Lint_Interval] := Module[{L},
   MonotoneEnclosure[(110 537 - 2290 L - 333 L^2 + 2 L^3)/(-49 052 + 314 027 L + 1024 L^2), L, Lint] - (kappa4[Lint] / π)/logrc[Lint]];
```

```
In[139]:= N[f[Linterval], 30]
Out[139]= Interval[{-0.616031369988227080395172238973, 0.710876622255693420794912548810}]
```

```
In[140]:= Timing[Length /@ ProveNonNegative[f, f, {Linterval}, MaxDepth → 20]]
Out[140]= {82.4985, {0, 0, 5453}}
```

Bound: $\frac{\kappa_5/\pi}{\mathrm{Log}[r/(c-1/2)]} \leq \frac{-2446-261\,L+200\,L^2-6\,L^3}{-231\,541-24\,108\,L+1024\,L^2}$

```
In[141]:= bound[9] = (-2446 - 261 L + 200 L^2 - 6 L^3)/(-231 541 - 24 108 L + 1024 L^2);
```

For the current values of c, r, we have $\theta_{-1/2} < \pi$. The quantity κ_5 is defined by

```
In[142]:= Clear[K5];
K5[c_, r_, η_] = 1/4 Block[{σ = c + r Cos[θ], t = r Sin[θ]}, Integrate[
    (1 - 2 σ) (2 t - 4 + 7/19 ((1 + σ)^2 + (t - 2)^2)), {θ, ArcCos[(-η - c)/r], ArcCos[(-1 / 2 - c)/r]}]]
```

Out[142]= $\frac{1}{76}\left(\frac{5}{2}\left(3+4c-4c^2-2r^2\right)-\frac{1}{2}r\sqrt{-\frac{1+4c+4c^2-4r^2}{r^2}}\left(-103+21c+28c^2+14r^2\right)+5\left(-2c+2c^2+r^2-2\eta\,(1+\eta)\right)+2r\left(-48+14c^2+7r^2-7c\,(-2+\eta)-7\eta\right)\sqrt{\frac{r^2-(c+\eta)^2}{r^2}}-\left(41+7r^2+c\left(-96+7c\,(3+2c)+28r^2\right)\right)\mathrm{ArcCos}\left[-\frac{\frac{1}{2}+c}{r}\right]+\left(41+7r^2+c\left(-96+7c\,(3+2c)+28r^2\right)\right)\mathrm{ArcCos}\left[-\frac{c+\eta}{r}\right]\right)$

Some easy rearranging work gives

In[143]:= κ_5[c_, r_, η_] := Block[{α1 = $\sqrt{r^2 - \left(c + \frac{1}{2}\right)^2}$, α2 = $\sqrt{r^2 - (c+\eta)^2}$},

$\frac{1}{76}\left(-\frac{5}{2}\left(-3 + 4(-1+c)\,c + 2r^2\right) - \alpha1\left(-103 + 7c(3+4c) + 14r^2\right) + 5\left(2(-1+c)\,c + r^2 - 2\eta(1+\eta)\right) + 2\left(-48 + 7r^2 + 7c(2+2c-\eta) - 7\eta\right)\alpha2 + \left(41 + 7r^2 + c\left(-96 + 7c(3+2c) + 28r^2\right)\right)\left(\text{ArcCos}\left[-\frac{c+\eta}{r}\right] - \text{ArcCos}\left[-\frac{\frac{1}{2}+c}{r}\right]\right)\right)$]

Mathematica is able to ascertain if the two expressions are equivalent, and they are.

In[144]:= Reduce[K5[c, r, η] == κ_5[c, r, η] && $-\frac{3}{2} < c - r < \frac{-1}{2} < 1 - c < 0 < 1 < c < c + r$]

Out[144]= $\left(\frac{3}{2} < r \le 2 \,\&\&\, 1 < c < \frac{1}{2}(-1+2r)\right) \,||\, \left(1 < c < \frac{3}{2} \,\&\&\, 2 < r \le \frac{5}{2}\right) \,||\, \left(\frac{5}{2} < r < 3 \,\&\&\, \frac{1}{2}(-3+2r) < c < \frac{3}{2}\right)$

In[145]:= % /. {c → c[L], r → r[L]}

Out[145]= $\left(\frac{3}{2} < \frac{149}{140} + \frac{747}{283+36L} \le 2 \,\&\&\, 1 < 1 + \frac{505}{430+111L} < \frac{1}{2}\left(-1 + 2\left(\frac{149}{140} + \frac{747}{283+36L}\right)\right)\right) \,||\, \left(1 < 1 + \frac{505}{430+111L} < \frac{3}{2} \,\&\&\, 2 < \frac{149}{140} + \frac{747}{283+36L} \le \frac{5}{2}\right) \,||\, \left(\frac{5}{2} < \frac{149}{140} + \frac{747}{283+36L} < 3 \,\&\&\, \frac{1}{2}\left(-3 + 2\left(\frac{149}{140} + \frac{747}{283+36L}\right)\right) < 1 + \frac{505}{430+111L} < \frac{3}{2}\right)$

In[146]:= Reduce[% && Lmin ≤ L ≤ Lmax]

Out[146]= $\frac{299}{50} \le L \le 28$

Now we turn to developing an interval enclosure for κ_5.

In[147]:=
```
(* the α1 is monotone decreasing, and so is its coefficient *)
Reduce[Lmin ≤ L ≤ Lmax && D[r[L]^2 - (c[L] + 1/2)^2, L] ≤ 0, L, Reals]
Reduce[Lmin ≤ L ≤ Lmax && D[(-103 + 7 c[L] (3 + 4 c[L]) + 14 r[L]^2), L] ≤ 0, L, Reals]
(* the α2 is monotone decreasing, and so is its coefficient *)
Reduce[Lmin ≤ L ≤ Lmax && D[r[L]^2 - (c[L] + η[L])^2, L] ≤ 0, L, Reals]
Reduce[
 Lmin ≤ L ≤ Lmax && D[2 (-48 + 7 r[L]^2 + 7 c[L] (2 + 2 c[L] - η[L]) - 7 η[L]), L] ≤ 0, L, Reals]
```

Out[147]= $\frac{299}{50} \leq L \leq 28$

Out[148]= $\frac{299}{50} \leq L \leq 28$

Out[149]= $\frac{299}{50} \leq L \leq 28$

Out[150]= $\frac{299}{50} \leq L \leq 28$

In[151]:=
```
(* the big terms are monotone increasing*)
Reduce[Lmin ≤ L ≤ Lmax && D[-5/2 (-3 + 4 (-1 + c[L]) c[L] + 2 r[L]^2) +
    5 (2 (-1 + c[L]) c[L] + r[L]^2 - 2 η[L] (1 + η[L])), L] ≥ 0, L, Reals]
```

Out[151]= $\frac{299}{50} \leq L \leq 28$

In[152]:=
```
(*the arccos coefficient is monotone decreasing *)
Reduce[Lmin ≤ L ≤ Lmax &&
  D[(41 + 7 r[L]^2 + c[L] (-96 + 7 c[L] (3 + 2 c[L]) + 28 r[L]^2)), L] ≤ 0, L, Reals]
(* the arccos term is monotone *)
Reduce[
 Lmin ≤ L ≤ Lmax && D[(ArcCos[-(c[L] + η[L])/r[L]] - ArcCos[-(1/2 + c[L])/r[L]]), L] ≤ 0, L, Reals]
```

Out[152]= $\frac{299}{50} \leq L \leq 28$

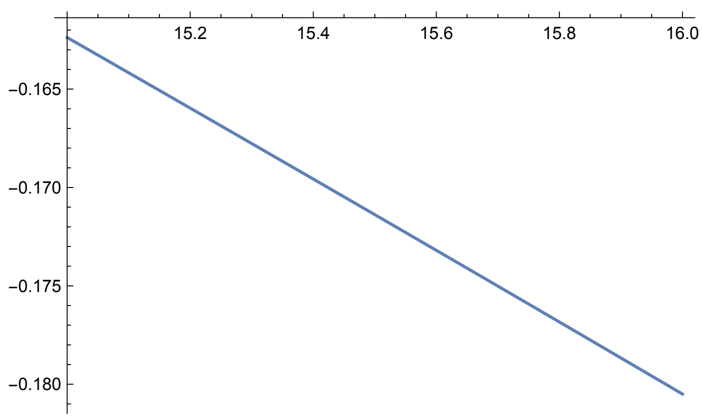
Out[153]= $\frac{299}{50} \leq L \leq 28$

In[154]:=
```
kappa5[L_] = κ5[c[L], r[L], η[L]];
kappa5[Lint_Interval] := Module[{a1c, α1, a2c, arccosc, arccos, L, bigs, α2},
  α1 = Sqrt[MonotoneEnclosure[r[L]^2 - (c[L] + 1/2)^2, L, Lint]];
  a1c = MonotoneEnclosure[(-103 + 7 c[L] (3 + 4 c[L]) + 14 r[L]^2), L, Lint];
  α2 = Sqrt[MonotoneEnclosure[r[L]^2 - (c[L] + η[L])^2, L, Lint]];
  a2c =
   MonotoneEnclosure[2 (-48 + 7 r[L]^2 + 7 c[L] (2 + 2 c[L] - η[L]) - 7 η[L]), L, Lint];
  arccos = MonotoneEnclosure[ArcCos[-(c[L] + η[L])/r[L]] - ArcCos[-(1/2 + c[L])/r[L]], L, Lint];
  arccosc = MonotoneEnclosure[
    (41 + 7 r[L]^2 + c[L] (-96 + 7 c[L] (3 + 2 c[L]) + 28 r[L]^2)), L, Lint];
  bigs = MonotoneEnclosure[-5/2 (-3 + 4 (-1 + c[L]) c[L] + 2 r[L]^2) +
     5 (2 (-1 + c[L]) c[L] + r[L]^2 - 2 η[L] (1 + η[L])), L, Lint];

  1/76 (bigs - a1c * α1 + a2c * α2 + arccosc * arccos)]
```

In[156]:=
```
Plot[kappa5[L], {L, 15, 16}]
N[kappa5[Interval[{15, 16}]], 20]
```

Out[156]=



Out[157]= Interval[{-0.26815176080293974618, -0.076672372310223473479}]

In[158]:=
```
Reduce[Lmin ≤ L ≤ Lmax && D[(-2446 - 261 L + 200 L^2 - 6 L^3)/(-231541 - 24108 L + 1024 L^2), L] ≤ 0, L, Reals]
```

Out[158]= $\frac{299}{50} \leq L \leq 28$

In[159]:= Clear[f];

$$f[L\_] = bound[9] - \frac{kappa5[L] / \pi}{logrc[L]};$$

f[Lint_Interval] := Module[{L},

$$MonotoneEnclosure\left[\frac{-2446 - 261 L + 200 L^2 - 6 L^3}{-231541 - 24108 L + 1024 L^2}, L, Lint\right] - \frac{kappa5[Lint] / \pi}{logrc[Lint]}];$$

In[162]:= N[f[Interval[{15 + 1/3, 16 - 1/3}]], 30]

Out[162]= Interval[{-0.00898181183027376217138717318809, 0.0137211016468456117417535230687}]

In[163]:= Timing[Length /@ ProveNonNegative[f, f, {Linterval}, MaxDepth → 25]]

Out[163]= {30.3143, {0, 0, 2896}}

Bound: $\frac{\kappa_{6,1} / \pi}{\mathrm{Log}[r/(c-1/2)]} \leq \frac{-1}{7} \mathrm{ArcCos}\left[\frac{L-6}{22}\right]$

In[164]:= $bound[10] = \frac{-1}{7} ArcCos\left[\frac{L-6}{22}\right];$

In[165]:= $Lstar[j\_, \theta\_] = 2 r Sin[\theta] - 4 + \frac{7}{19} \left((j + c + r Cos[\theta])^2 + (r Sin[\theta] - 2)^2\right);$

In[166]:= Clear[K61];

$$K61[c\_, r\_] = FullSimplify\Big[\frac{1}{4} Block\Big[\{\sigma = c + r Cos[\theta], t = r Sin[\theta]\},$$

$$Integrate\Big[(1 - 2\sigma - 2) Lstar[2, \theta] + 2 Lstar[0, \theta], \Big\{\theta, ArcCos\Big[\frac{-1/2 - c}{r}\Big], \pi\Big\}\Big]\Big],$$

$$-\frac{3}{2} < c - r < \frac{-1}{2} < 1 - c < 0 < 1 < c < c + r\Big]$$

Out[167]=

$$\frac{1}{76}\left(\frac{5}{2} + 10 c (1 + c) + 5 r^2 + 5 r (-2 - 4 c + r) + \frac{1}{2}\sqrt{-1 - 4 c (1 + c) + 4 r^2}\right.$$
$$(-26 + 7 c (13 + 4 c) + 14 r^2) - \pi (-20 - 12 c + 63 c^2 + 14 c^3 + 7 (5 + 4 c) r^2) +$$
$$\left.(-20 - 12 c + 63 c^2 + 14 c^3 + 7 (5 + 4 c) r^2) ArcCos\left[-\frac{\frac{1}{2} + c}{r}\right]\right) +$$
$$\frac{1}{38}\left(-5 + 10 r - c \left(10 + 7\sqrt{-1 - 4 c (1 + c) + 4 r^2}\right) + (-48 + 7 c^2 + 7 r^2) ArcCos\left[\frac{\frac{1}{2} + c}{r}\right]\right)$$

Some easy rearranging work gives

In[168]:=
```
κ6[c_, r_] := Block[{α1 = Sqrt[r^2 - (c + 1/2)^2]},
  1/152 (5 (-3 + 2 c - 2 r) (1 + 2 c - 2 r) + 2 (-26 + 7 c (9 + 4 c) + 14 r^2) α1 -
    2 (76 - 12 c + 49 c^2 + 14 c^3 + 7 (3 + 4 c) r^2) ArcCos[(1/2 + c)/r])]
```

In[169]:=
```
Reduce[K61[c, r] == κ6[c, r] && -3/2 < c - r < -1/2 < 1 - c < 0 < 1 < c < c + r]
```

Out[169]= $1 < c < \frac{3}{2}$ && $\frac{1}{2}(1 + 2c) < r < \frac{1}{2}(3 + 2c)$

In[170]:=
```
Reduce[1 < c[L] < 3/2 && 1/2 (1 + 2 c[L]) < r[L] < 1/2 (3 + 2 c[L]) && Lmin ≤ L ≤ Lmax]
Lmin ≤ L ≤ Lmax
```

Out[170]= $\frac{299}{50} \le L \le 28$

Out[171]= $\frac{299}{50} \le L \le 28$

In[172]:=
```
(* Everything in sight is monotonic *)
Reduce[Lmin ≤ L ≤ Lmax && D[5 (-3 + 2 c[L] - 2 r[L]) (1 + 2 c[L] - 2 r[L]), L] == 0]
Reduce[Lmin ≤ L ≤ Lmax && D[r[L]^2 - (c[L] + 1/2)^2, L] == 0]
Reduce[Lmin ≤ L ≤ Lmax && D[(-26 + 7 c[L] (9 + 4 c[L]) + 14 r[L]^2), L] == 0]
Reduce[
 Lmin ≤ L ≤ Lmax && D[(76 - 12 c[L] + 49 c[L]^2 + 14 c[L]^3 + 7 (3 + 4 c[L]) r[L]^2), L] == 0]
Reduce[Lmin ≤ L ≤ Lmax && D[(1/2 + c[L])/r[L], L] == 0]
```

Out[172]= False

Out[173]= False

Out[174]= False

Out[175]= False

Out[176]= False

In[177]:=
```
kappa61[L_] = κ6[c[L], r[L]];
kappa61[Lint_Interval] := Module[{p1, α1, a1c, arccosc, arccos, L},
  p1 = MonotoneEnclosure[5 (-3 + 2 c[L] - 2 r[L]) (1 + 2 c[L] - 2 r[L]), L, Lint];
  α1 = MonotoneEnclosure[Sqrt[r[L]^2 - (c[L] + 1/2)^2], L, Lint];
  a1c = MonotoneEnclosure[2 (-26 + 7 c[L] (9 + 4 c[L]) + 14 r[L]^2), L, Lint];
  arccosc = MonotoneEnclosure[
    2 (76 - 12 c[L] + 49 c[L]^2 + 14 c[L]^3 + 7 (3 + 4 c[L]) r[L]^2) , L, Lint];
  arccos = MonotoneEnclosure[ArcCos[(1/2 + c[L])/r[L]], L, Lint];

  1/152 (p1 + a1c * α1 - arccosc * arccos)]
```

In[179]:=
```
Clear[f];
f[L_] := -1/7 ArcCos[(L - 6)/22] - (kappa61[L] / π)/logrc[L];
```

In[181]:=
```
Timing[Length /@ ProveNonNegative[f, f, {Linterval}, MaxDepth → 25]]
```

Out[181]= {1.62998, {0, 0, 264}}

Assembling the final bound

In[182]:=
```
sharp[L_, T_] = Sum[bound[k], {k, 7}] + 1/(T + 2) Sum[bound[k], {k, 8, 10}]
```

Out[182]=
$$\frac{1}{1024}+\frac{-6171+827\,L+537\,L^2}{-92\,820+31\,635\,L+1024\,L^2}+\frac{16\,489+14\,928\,L+402\,L^2}{380\,433+35\,739\,L+1024\,L^2}+\frac{-82\,709+17\,494\,L+3466\,L^2-2\,L^3}{2\left(-188\,287+62\,894\,L+512\,L^2\right)}+\frac{35\,688+11\,843\,L+622\,L^2+L^3}{689\,229+58\,176\,L+1024\,L^2}+\frac{199\,847+79\,917\,L+5034\,L^2+7\,L^3}{2\left(192\,512+31\,343\,L+512\,L^2\right)}+\frac{62\,044+305\,799\,L+18\,763\,L^2+235\,L^3}{2\left(243\,814+38\,507\,L+512\,L^2\right)}+\frac{6}{55\left(-\frac{1}{5}+T\right)}+\frac{\frac{-2446-261\,L+200\,L^2-6\,L^3}{-231\,541-24\,108\,L+1024\,L^2}+\frac{110\,537-2290\,L-333\,L^2+2\,L^3}{-49\,052+314\,027\,L+1024\,L^2}-\frac{1}{7}\,\mathrm{ArcCos}\left[\frac{1}{22}\left(-6+L\right)\right]}{2+T}$$

First we wrestle with the terms involving *T*.

In[183]:=
```
Clear[f];
f[L_, T_] = (6/55)/(T - 1/5) + 1/(T + 2) (bound[8] + bound[9] + bound[10])
```

Out[184]=
$$\frac{6}{55\left(-\frac{1}{5}+T\right)}+\frac{\frac{-2446-261\,L+200\,L^2-6\,L^3}{-231\,541-24\,108\,L+1024\,L^2}+\frac{110\,537-2290\,L-333\,L^2+2\,L^3}{-49\,052+314\,027\,L+1024\,L^2}-\frac{1}{7}\,\mathrm{ArcCos}\left[\frac{1}{22}\left(-6+L\right)\right]}{2+T}$$

In[185]:=
```
Reduce[Lmin ≤ L ≤ Lmax && D[bound[8] + bound[9] + bound[10], L] == 0]
```
Out[185]=
```
L == 17.9...
```

That is, f is unimodal in *L*. It is not monotone in *T*, but no matter.

In[186]:=
```
f[Lint_Interval, Tint_Interval] := Module[{L, Aint},
  Aint = PiecewiseMonotoneEnclosure[(110537 - 2290 L - 333 L^2 + 2 L^3)/(-49052 + 314027 L + 1024 L^2) +
     (-2446 - 261 L + 200 L^2 - 6 L^3)/(-231541 - 24108 L + 1024 L^2) + (-1)/7 ArcCos[(L - 6)/22], L, {17.9...}, Lint];
  (6/55)/(Tint - 1/5) + Aint/(2 + Tint)]
```

$$f[\text{Lint\_Interval}, \text{Tint\_Interval}] := \text{Module}\Big[\{L, \text{Aint}\}, \text{Aint} = \text{PiecewiseMonotoneEnclosure}\Big[\frac{110537 - 2290 L - 333 L^2 + 2 L^3}{-49052 + 314027 L + 1024 L^2} + \frac{-2446 - 261 L + 200 L^2 - 6 L^3}{-231541 - 24108 L + 1024 L^2} + \frac{-1}{7}\text{ArcCos}\Big[\frac{L-6}{22}\Big], L, \{17.9\ldots\}, \text{Lint}\Big]; \frac{6/55}{\text{Tint} - 1/5} + \frac{\text{Aint}}{2 + \text{Tint}}\Big]$$

In[187]:=
```
f[Interval[{15, 16}], Interval[{5/7, ∞}]]
```
Out[187]=

$$\text{Interval}\Big[\Big\{\frac{7}{19}\Big(-\frac{105882794923}{1859975710500} - \frac{1}{7}\text{ArcCos}\Big[\frac{5}{11}\Big]\Big), \frac{7}{33}\Big\}\Big]$$

In[188]:=

$$\text{tobeprovennonneg}[L\_, T\_] := \frac{13}{86} - f[L, T];$$

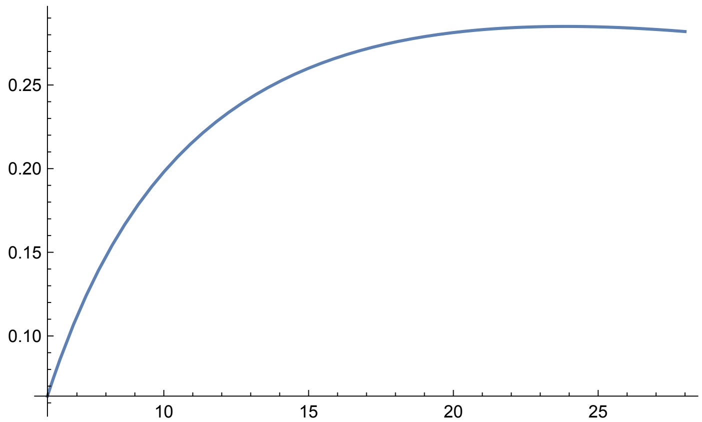
In[189]:=
```
ProveNonNegative[tobeprovennonneg,
 tobeprovennonneg, {Linterval, Interval[{5/7, ∞}]}]
```
Out[189]=

$$\{\{\}, \{\}, \{\{\text{Interval}[\{\tfrac{299}{50}, 16\}], \text{Interval}[\{\tfrac{5}{7}, 1\}]\}, \{\text{Interval}[\{\tfrac{299}{50}, 16\}], \text{Interval}[\{1, \infty\}]\},$$
$$\{\text{Interval}[\{16, 24\}], \text{Interval}[\{\tfrac{5}{7}, \tfrac{3}{4}\}]\}, \{\text{Interval}[\{16, 24\}], \text{Interval}[\{\tfrac{3}{4}, 1\}]\},$$
$$\{\text{Interval}[\{24, 26\}], \text{Interval}[\{\tfrac{5}{7}, \tfrac{23}{32}\}]\}, \{\text{Interval}[\{24, 26\}], \text{Interval}[\{\tfrac{23}{32}, \tfrac{3}{4}\}]\},$$
$$\{\text{Interval}[\{26, 27\}], \text{Interval}[\{\tfrac{5}{7}, \tfrac{183}{256}\}]\},$$
$$\{\text{Interval}[\{26, 27\}], \text{Interval}[\{\tfrac{183}{256}, \tfrac{23}{32}\}]\},$$
$$\{\text{Interval}[\{27, 28\}], \text{Interval}[\{\tfrac{5}{7}, \tfrac{183}{256}\}]\},$$
$$\{\text{Interval}[\{27, 28\}], \text{Interval}[\{\tfrac{183}{256}, \tfrac{23}{32}\}]\},$$
$$\{\text{Interval}[\{26, 28\}], \text{Interval}[\{\tfrac{23}{32}, \tfrac{3}{4}\}]\},$$
$$\{\text{Interval}[\{24, 28\}], \text{Interval}[\{\tfrac{3}{4}, 1\}]\}, \{\text{Interval}[\{16, 28\}], \text{Interval}[\{1, \infty\}]\}\}\}$$

In[190]:=
$$\text{sharp2}[L\_] = \frac{1}{1024} + \frac{-6171 + 827\,L + 537\,L^2}{-92\,820 + 31\,635\,L + 1024\,L^2} + \frac{16\,489 + 14\,928\,L + 402\,L^2}{380\,433 + 35\,739\,L + 1024\,L^2} + \frac{-82\,709 + 17\,494\,L + 3466\,L^2 - 2\,L^3}{2\,(-188\,287 + 62\,894\,L + 512\,L^2)} + \frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2} + \frac{199\,847 + 79\,917\,L + 5034\,L^2 + 7\,L^3}{2\,(192\,512 + 31\,343\,L + 512\,L^2)} + \frac{62\,044 + 305\,799\,L + 18\,763\,L^2 + 235\,L^3}{2\,(243\,814 + 38\,507\,L + 512\,L^2)} + \frac{13}{86};$$

In[191]:=
$$\text{Plot}\left[0.22737\,L + 2\,\text{Log}[1 + L] - \frac{1}{2} - \text{sharp2}[L],\ \{L, \text{Lmin}, \text{Lmax}\},\ \text{PlotRange} \to \text{All}\right]$$

Out[191]=



In[192]:=
$$\text{toprovenonneg}[L\_] = \frac{22\,737}{100\,000}\,L + 2\,\text{Log}[1 + L] - \frac{1}{2} - \text{sharp2}[L]$$

Out[192]=
$$-\frac{28\,715}{44\,032} + \frac{22\,737\,L}{100\,000} - \frac{-6171 + 827\,L + 537\,L^2}{-92\,820 + 31\,635\,L + 1024\,L^2} - \frac{16\,489 + 14\,928\,L + 402\,L^2}{380\,433 + 35\,739\,L + 1024\,L^2} - \frac{-82\,709 + 17\,494\,L + 3466\,L^2 - 2\,L^3}{2\,(-188\,287 + 62\,894\,L + 512\,L^2)} - \frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2} - \frac{199\,847 + 79\,917\,L + 5034\,L^2 + 7\,L^3}{2\,(192\,512 + 31\,343\,L + 512\,L^2)} - \frac{62\,044 + 305\,799\,L + 18\,763\,L^2 + 235\,L^3}{2\,(243\,814 + 38\,507\,L + 512\,L^2)} + 2\,\text{Log}[1 + L]$$

The rational part of toprovenonneg[L] is monotone decreasing:

In[193]:=
$$\text{Reduce}\Big[L \geq \text{Lmin}\ \&\&\ D\Big[-\frac{28\,715}{44\,032} + \frac{22\,737\,L}{100\,000} - \frac{-6171 + 827\,L + 537\,L^2}{-92\,820 + 31\,635\,L + 1024\,L^2} - \frac{16\,489 + 14\,928\,L + 402\,L^2}{380\,433 + 35\,739\,L + 1024\,L^2} - \frac{-82\,709 + 17\,494\,L + 3466\,L^2 - 2\,L^3}{2\,(-188\,287 + 62\,894\,L + 512\,L^2)} - \frac{35\,688 + 11\,843\,L + 622\,L^2 + L^3}{689\,229 + 58\,176\,L + 1024\,L^2} - \frac{199\,847 + 79\,917\,L + 5034\,L^2 + 7\,L^3}{2\,(192\,512 + 31\,343\,L + 512\,L^2)} - \frac{62\,044 + 305\,799\,L + 18\,763\,L^2 + 235\,L^3}{2\,(243\,814 + 38\,507\,L + 512\,L^2)},\ L\Big] \leq 0\Big]$$

Out[193]=
$$L \geq \frac{299}{50}$$

In[194]:=
```
toprovenonneg[Lint_Interval] := Module[{L, Lmin, Lmax, expr, min, max},
  {Lmin, Lmax} = {Min[Lint], Max[Lint]};
  expr = -28715/44032 + 22737 L/100000 - (-6171 + 827 L + 537 L^2)/(-92820 + 31635 L + 1024 L^2) - (16489 + 14928 L + 402 L^2)/(380433 + 35739 L + 1024 L^2) -
    (-82709 + 17494 L + 3466 L^2 - 2 L^3)/(2 (-188287 + 62894 L + 512 L^2)) - (35688 + 11843 L + 622 L^2 + L^3)/(689229 + 58176 L + 1024 L^2) -
    (199847 + 79917 L + 5034 L^2 + 7 L^3)/(2 (192512 + 31343 L + 512 L^2)) - (62044 + 305799 L + 18763 L^2 + 235 L^3)/(2 (243814 + 38507 L + 512 L^2));
  exprint = MonotoneEnclosure[expr, L, Lint];
  exprint + 2 Log[1 + Lint]];
```

In[195]:=
```
N[toprovenonneg[Linterval], 30]
```

Out[195]=
```
Interval[{-2.56656466726227320137609424733, 2.91253178932518202378775018011}]
```

In[196]:=
```
ProveNonNegative[toprovenonneg, toprovenonneg, {Linterval}, MaxDepth → 15]
```

Out[196]=
```
{{}, {}, {{Interval[{299/50, 6}]}, {Interval[{6, 25/4}]}, {Interval[{25/4, 13/2}]},
  {Interval[{13/2, 27/4}]}, {Interval[{27/4, 7}]}, {Interval[{7, 15/2}]},
  {Interval[{15/2, 8}]}, {Interval[{8, 17/2}]}, {Interval[{17/2, 9}]},
  {Interval[{9, 10}]}, {Interval[{10, 11}]}, {Interval[{11, 12}]},
  {Interval[{12, 13}]}, {Interval[{13, 14}]}, {Interval[{14, 16}]},
  {Interval[{16, 18}]}, {Interval[{18, 20}]}, {Interval[{20, 22}]},
  {Interval[{22, 24}]}, {Interval[{24, 26}]}, {Interval[{26, 28}]}}}
```